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Characteristic subgroups in locally finite groups

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ABSTRACT

Let G be a locally finite group having a subgroup N of finite index n that possesses a normal series

$$N = N_0 \geqslant N_1 \geqslant \cdots \geqslant N_k = 1$$

each of whose quotients N_i/N_{i+1} is either locally nilpotent or satisfies an outer commutator law $w_i \equiv 1$. We show that G contains a characteristic subgroup H of finite index that has a characteristic series with the same properties. Moreover, the index of H in G is bounded by a function depending only on n, k and the weight of the w_i.

1. Introduction

Let G be a group that has a subgroup N of finite index with certain property P. Recently some attention was given to the question whether G necessarily contains a characteristic subgroup of finite index with the property P. Since any finite-index subgroup $N < G$ contains a normal in G subgroup of finite (dividing $|G : N|!$) index, in many instances we can assume that N is normal in G. If G is

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finitely generated, then N contains a fully invariant (and even verbal) in G subgroup of finite index (which depends on the number of generators). Therefore, for finitely generated groups the answer to the above question is positive whenever P is inherited by subgroups.

For arbitrary groups it is known that the answer is positive for example for the following properties:

– P is "abelian"; this fact is known as Passman's lemma (see Chapter 12, Lemma 1.2 in [1] or Lemma 21.1.4 in [2]);
– P is "nilpotent" or "locally nilpotent", or "solvable", or "locally finite" (consider the automorphic closure $\prod_{\alpha \in \mathrm{Aut}\, G} N^{\alpha}$, which is a characteristic P-subgroup);
– P is "nilpotent of given class c" (see Lemma 3 in [3]);
– P is "satisfying a given outer (multilinear) commutator law w" (original proof can be found in [4], other shorter proofs are given in [5,6]).

It is also known that if P is "periodic of exponent p", then the answer is negative for large primes p (see an example in [6]).

In the present paper we consider the property for N to have a normal series

$$N = N_0 \geqslant N_1 \geqslant \cdots \geqslant N_k = 1$$

of finite length k each of whose quotients N_i/N_{i+1} is either locally nilpotent or satisfies an outer commutator law.

Our main result is the following theorem.

Theorem 1.1. *Let G be a locally finite group that contains a normal subgroup N of finite index n that possesses a normal series*

$$N = N_0 \geqslant N_1 \geqslant \cdots \geqslant N_k = 1$$

each of whose quotients N_i/N_{i+1} is either locally nilpotent or satisfies an outer commutator law $w_i \equiv 1$. Then G contains a finite-index characteristic subgroup H that has a characteristic series

$$H = H_0 \geqslant H_1 \geqslant \cdots \geqslant H_k = 1$$

whose quotients H_i/H_{i+1} have the same properties as the quotients N_i/N_{i+1}, that is, if N_i/N_{i+1} is locally nilpotent, then so is H_i/H_{i+1}, and if N_i/N_{i+1} satisfies an outer commutator identity $\omega \equiv 1$, then so does H_i/H_{i+1}. Moreover, the index of H in G is bounded by a function depending only on n, k and the weights of the w_i.

We mention that the particular case of the above theorem where all quotients N_i/N_{i+1} satisfy outer commutator laws follows easily from the results obtained earlier in [4] (also in [5] or [6]). On the other hand, the case where all quotients N_i/N_{i+1} are locally nilpotent follows from the Hirsch–Plotkin theorem that in any group a product of normal locally nilpotent subgroups is again locally nilpotent. So it is a combination of outer commutator laws and local nilpotency, where Theorem 1.1 gives a new result. We do not know whether the theorem remains valid if the assumption that G is locally finite is removed from the hypothesis.

2. Star words and corresponding subgroups

If X, Y are subsets of a periodic group G, we denote by $[X, Y]_*$ the subgroup generated by all the commutators $[x, y]$, where $x \in X$, $y \in Y$ and the elements x, y have coprime orders, that is

$$[X, Y]_* = \big\langle [x, y];\ \text{where } x \in X,\ y \in Y \text{ and } \big(|x|, |y|\big) = 1\big\rangle. \tag{1}$$

Obviously, if X and Y are normal (characteristic) subgroups then the subgroup $[X,Y]_*$ is normal (characteristic) as well. It is easy to see that a locally finite group G is locally nilpotent if and only if $[G,G]_* = 1$.

Lemma 2.1. *If G is a locally finite group, then $[G,G]_*$ is the locally nilpotent residual of G.*

Proof. We need to prove that G/A is locally nilpotent for some normal subgroup A if and only if $[G,G]_* \leqslant A$. If G/A is locally nilpotent, it is obvious that $[G,G]_* \leqslant A$. Let now $[G,G]_* \leqslant A$. We take two elements $x, y \in G$ such that their images $\bar{x}$, $\bar{y}$ in the quotient G/A have prime-power orders for different primes: $|\bar{x}| = p^r$ and $|\bar{y}| = q^s$. Let $|x| = p^i n$ with $(n,p) = 1$ and $|y| = q^j m$ with $(q,m) = 1$. By (1) we have $[x^n, y^m] \in [G,G]_*$, so $[x^n, y^m] \in A$. We choose the positive integers k and l such that $nk \equiv 1 (\mathrm{mod}\ p^i)$ and $ml \equiv 1 (\mathrm{mod}\ q^j)$. Then

$$1 \equiv [x^n, y^m] \equiv \left[(x^n)^k, (y^m)^l\right] \equiv [x,y] (\mathrm{mod}\ A).$$

(Here we use the property that if $[a,b] \in N$, then $[a^k, b^l] \in N$ for any normal subgroup N and any positive integers k, l.) Thus, any two elements of prime-power orders for different primes commute in the quotient G/A and therefore G/A is locally nilpotent. □

Recall that outer commutator words are group words obtained by nesting commutators using always different variables. Thus $[[x_1,x_2],[x_3,x_4,x_5],x_6]$ is an outer commutator word, but the Engel word $[x_1,x_2,x_2,x_2]$ is not.

The concept of a star word can be introduced as follows. Let $x_1, x_2, \ldots$ be variables. A *star word of weight* 1 is just a variable x_i. A *star word of weight* $t > 1$ is either a formal expression of the form $W(x_1,\ldots,x_t) = [U(x_1,\ldots,x_r), V(x_{r+1},\ldots,x_t)]$ or $W(x_1,\ldots,x_t) = [U(x_1,\ldots,x_r), V(x_{r+1},\ldots,x_t)]_*$, where U and V are star words of weights r and $t-r$ respectively.

If G is a group and w a group word, the verbal subgroup $w(G)$ corresponding to the word w is the subgroup generated by all w-values in G. If W is a star word, we can define the corresponding subgroup $W(G)$ of G in the following way. If W is of weight 1, then $W(G)$ is precisely G. Suppose that W is of weight $t \geqslant 2$ and assume by induction that subgroups corresponding to star words of weight at most $t-1$ are already defined. If

$$W(x_1,\ldots,x_t) = \left[U(x_1,\ldots,x_r), V(x_{r+1},\ldots,x_t)\right],$$

then $W(G) = [U(G), V(G)]$. If

$$W(x_1,\ldots,x_t) = \left[U(x_1,\ldots,x_r), V(x_{r+1},\ldots,x_t)\right]_*,$$

then $W(G) = [U(G), V(G)]_*$ is the subgroup defined in (1). A little more generally, if $W = W(x_1,\ldots,x_t)$ and $X_1,\ldots,X_t$ are subsets of G, the symbol $W(X_1,\ldots,X_t)$ stands for the corresponding subgroup of G. In the particular case where $X_1 = X_2 = \cdots = X_t = G$ we obviously have

$$W(\underbrace{G,\ldots,G}_{t}) = W(G).$$

Thus, every star word can be treated as a function defined on the set of all subgroups of any locally finite group G. Note that if H is a normal (characteristic) subgroup of a group G then $W(H)$ is also a normal (characteristic) subgroup.

If $W_1 = W_1(x_1,\ldots,x_t)$ and $W_2 = W_2(y_1,\ldots,y_s)$ are two star words, we denote by $W_1 \circ W_2$ the star word

$$W_1\big(W_2(y_{11},\ldots,y_{s1}),\ldots,W_2(y_{1t},\ldots,y_{st})\big).$$

By induction we define $W_1 \circ W_2 \circ \cdots \circ W_n = (W_1 \circ W_2 \circ \cdots \circ W_{n-1}) \circ W_n$.

It is easy to see that the subgroup $(W_1 \circ W_2)(G)$ corresponding to the star word $(W_1 \circ W_2)$ is equal to the subgroup $W_1(W_2(G))$. We also mention that the operation $\circ$ is associative on the set of star words, that is

$$(W_1 \circ W_2) \circ W_3 = W_1 \circ (W_2 \circ W_3).$$

Lemma 2.2. *A locally finite group G has a normal series*

$$G = G_0 \geqslant G_1 \geqslant \cdots \geqslant G_k = 1$$

each of whose quotients G_i/G_{i+1} is either locally nilpotent or satisfies an outer commutator law $w_i \equiv 1$ if and only if $W(G) = 1$, where $W = W_{k-1} \circ \cdots \circ W_1 \circ W_0$ with $W_i = w_i$ if the quotient G_i/G_{i+1} satisfies the outer commutator law $w_i \equiv 1$, and $W_i = [x, y]_*$ if the quotient G_i/G_{i+1} is locally nilpotent.

Proof. This is easy by induction on k. Suppose that G possesses the above series. If $k \leqslant 1$, the result follows from Lemma 2.1. Assume that $k \geqslant 2$ and set $U = W_{k-1} \circ \cdots \circ W_2 \circ W_1$. By induction $U(G_1) = 1$. It is clear that $W_0(G) \leqslant G_1$ so $W(G) = (U \circ W_0)(G) = U(W_0(G)) = 1$, as required.

If $W(G) = 1$, then the required normal series is

$$G = G_0 \geqslant G_1 \geqslant \cdots \geqslant G_k = 1,$$

where $G_i = W_{i-1}(W_{i-2}(G)) = (W_{i-1} \circ \cdots \circ W_1 \circ W_0)(G)$. $\square$

3. Proof of the main result

The next lemma plays a crucial role for the proof of Theorem 1.1.

Lemma 3.1. *Let G be a locally finite group. If A, B, Y are normal subgroups of G, then $[AB, Y]_* = [A, Y]_*[B, Y]_*$.*

Proof. The inclusion $[AB, Y]_* \supseteq [A, Y]_*[B, Y]_*$ is obvious, so we prove $[AB, Y]_* \subseteq [A, Y]_*[B, Y]_*$. Let A, B be normal subgroups of G. Choose an element $h \in [AB, Y]_*$ and write

$$h = [x_1, y_1] \cdots [x_n, y_n],$$

where $x_1, \ldots, x_n \in AB$, $y_1, \ldots, y_n \in Y$ and $(|x_i|, |y_i|) = 1$ for all i.

We represent every x_i as a product $a_{i_1} b_{i_1} a_{i_2} b_{i_2} \cdots a_{i_k} b_{i_k}$ of some elements $a_{i_j} \in A$, $b_{i_j} \in B$ such that $(|a_{i_j}|, |y_i|) = 1$ and $(|b_{i_j}|, |y_i|) = 1$. This is always possible. Indeed, let $x_i = a_i b_i$, where $a_i \in A$, $b_i \in B$, and $|x_i| = p_1^{s_1} \cdots p_t^{s_t}$ for some primes $p_1, p_2, \ldots, p_t$. For every $1 \leqslant j \leqslant t$ we set $n_j = p_1^{s_1} \cdots p_{j-1}^{s_{j-1}} p_{j+1}^{s_{j+1}} \cdots p_t^{s_t}$. The element $x_i^{n_j}$ is a p_j-element and belongs to some p_j-Sylow subgroup of the finite group $H = \langle a, b \rangle$. We have $H = \langle a \rangle^H \langle b \rangle^H$ and use the fact that a Sylow subgroup of a product of normal subgroups is equal to a product of Sylow subgroups of the factors. Therefore $x_i^{n_j} = a_{p_j} b_{p_j}$ for some p_j-elements $a_{p_j} \in \langle a \rangle^H \leqslant A$, $b_{p_j} \in \langle b \rangle^H \leqslant B$. There are integers $f_1, \ldots, f_t$ such that $f_1 n_1 + f_2 n_2 + \cdots + f_t n_t = 1$. So

$$x_i = \prod_{j=1}^{t} (a_{p_j} b_{p_j})^{f_j},$$

and we get the required representation of x_i.

Using repeatedly the identity $[ab,c]=[a^b,c^b][b,c]$ we obtain that the commutator $[x_i,y_i]=[a_{i_1}b_{i_1}a_{i_2}\cdots b_{i_k},y_i]$ is equal to a product of some conjugates of the commutators $[a_{i_1},y_i],[b_{i_1},y_i],\dots,[b_{i_k},y_i]$.

Taking into consideration that A, B are normal and $(|a_{i_j}|,|y_i|)=1$, $(|b_{i_j}|,|y_i|)=1$ we obtain that $[a^g_{i_j},y^g_i]\in[A,Y]_*$, $[b^g_{i_j},y^g_i]\in[B,Y]_*$ for all $g\in G$ and for all j. Thus, the element $h=[x_1,y_1]\cdots[x_n,y_n]$ is the product of some commutators from $[A,Y]_*$ and $[B,Y]_*$. Since the subgroups $[A,Y]_*$ and $[X,Y]_*$ are normal, $h\in[A,Y]_*[B,Y]_*$. □

Corollary 3.2. *Let W be a star word of weight t, and $A_1,\dots,A_t$ normal subgroups of a group G.*

$$W\left(A_1,\dots,A_{i-1},\prod_{N\in\mathcal{N}}N,A_{i+1},\dots,A_t\right)=\prod_{N\in\mathcal{N}}W(A_1,\dots,A_{i-1},N,A_{i+1},\dots,A_t)$$

for any set $\mathcal{N}$ of normal subgroups.

Proof. We expand $W(A_1,\dots,A_{i-1},\prod_{N\in\mathcal{N}}N,A_{i+1},\dots,A_t)$ by applying repeatedly the equalities $[A,BC]=[A,B][A,C]$ and $[A,BC]_*=[A,B]_*[A,C]_*$. This is possible because all the "star-commutator subgroups" arising in this process are normal. In the end we obtain the product

$$\prod_{N\in\mathcal{N}}W(A_1,\dots,A_{i-1},N,A_{i+1},\dots,A_t). \qquad \square$$

In what follows $W_\sigma(X_1,\dots,X_t)$ denotes $W(X_{\sigma(1)},\dots,X_{\sigma(t)})$, where σ is a permutation of degree t.

Theorem 3.3. *Let $W(x_1,\dots,x_t)$ be a star word. Then, in any group, the number of finite-index subgroups which are maximal* (*by inclusion*) *among all normal subgroups N such that $W(N)=1$ is finite. Moreover, the number of such subgroups of index $\leqslant n$ does not exceed*

$$2^{F^{t-1}(n)},\ \text{where } F^k(x) \text{ is the } k\text{-th iteration of the function } F(x)=xn^{2^x}.$$

Proof. The proof mimicks that of Theorem 1′ in [6]. We include it for completeness. Let $\mathcal{N}$ be the set of finite-index subgroups N of G that are maximal by inclusion among all normal subgroups satisfying the property $W(N)=1$. We wish to prove that $\mathcal{N}$ is finite. If the set $\mathcal{N}$ is empty, then we have nothing to prove. Otherwise, consider a subgroup $G_0\in\mathcal{N}$. This subgroup satisfies the property that

$$W_\sigma(G_0,\dots,G_0)=W(G_0)=1 \quad \text{for all } \sigma\in S_t, \tag{2}$$

where S_t is the symmetric group of degree t.

The subgroup G_0 has finite index. Therefore, the set of subgroups $\{NG_0 \mid N\in\mathcal{N}\}$ is finite and coincides with the set $\{N_1G_0 \mid N_1\in\mathcal{N}_1\}$, where $\mathcal{N}_1$ is some finite subset of $\mathcal{N}$. The subgroup

$$G_1=G_0\cap\bigcap_{N\in\mathcal{N}_1}N$$

has finite index and satisfies the equality

$$W_\sigma(G_1,\dots,G_1,NG_0)=1 \quad \text{for all } \sigma\in S_t \text{ and for all } N\in\mathcal{N}. \tag{3}$$

Indeed, by the choice of $\mathcal{N}_1$, each product NG_0, where $N\in\mathcal{N}$, coincides with a product N_1G_0 for some group $N_1\in\mathcal{N}_1$ and $N_1\supseteq G_1\subseteq G_0$. Therefore, by Corollary 3.2

$$\begin{aligned} W_\sigma(G_1,\dots,G_1,NG_0) &= W_\sigma(G_1,\dots,G_1,N_1G_0) \\ &\subseteq W_\sigma(G_1,\dots,G_1,N_1)W_\sigma(G_1,\dots,G_1,G_0) \\ &\subseteq W_\sigma(N_1,\dots,N_1,N_1)W_\sigma(G_0,\dots,G_0,G_0) = 1. \end{aligned}$$

The subgroup G_1 has finite index. Therefore, the set of subgroups $\{NG_1 \mid N \in \mathcal{N}\}$ is finite and coincides with the set $\{N_2G_1 \mid N_2 \in \mathcal{N}_2\}$ for some finite subset $\mathcal{N}_2$ of $\mathcal{N}$. The subgroup

$$G_2 = G_1 \cap \bigcap_{N\in\mathcal{N}_2} N$$

has finite index and satisfies the equality

$$W_\sigma(G_2,\dots,G_2,NG_1,NG_1) = 1 \quad \text{for all } \sigma \in S_t \text{ and for all } N \in \mathcal{N}.$$

Indeed, by the choice of $\mathcal{N}_2$, each product NG_1, where $N \in \mathcal{N}$, coincides with a product N_2G_1 for some group $N_2 \in \mathcal{N}_2$ and $N_2 \supseteq G_2 \subseteq G_1 \subseteq G_0$. Therefore, by Corollary 3.2

$$\begin{aligned} W_\sigma(G_2,\dots,G_2,NG_1,NG_1) &= W_\sigma(G_2,\dots,G_2,N_2G_1,N_2G_1) \\ &= W_\sigma(G_2,\dots,G_2,N_2,N_2)W_\sigma(G_2,\dots,G_2,N_2,G_1) \\ &\quad \times W_\sigma(G_2,\dots,G_2,G_1,N_2)W_\sigma(G_2,\dots,G_2,G_1,G_1) \\ &\subseteq W_\sigma(N_2,\dots,N_2,N_2,N_2)W_\sigma(G_1,\dots,G_1,N_2,G_1) \\ &\quad \times W_\sigma(G_1,\dots,G_1,G_1,N_2)W_\sigma(G_0,\dots,G_0,G_0,G_0). \end{aligned}$$

The first factor of the last product is trivial, because the group N_2 satisfies the property $W(N_2) = 1$. The second and the third factors are trivial by (3). The fourth factor is trivial by (2).

Continuing in the same manner, we finally obtain a finite-index subgroup G_{t-1} such that

$$W_\sigma(NG_{t-1},\dots,NG_{t-1}) = 1 \quad \text{for all } \sigma \in S_t \text{ and for all } N \in \mathcal{N}.$$

In view of the maximality of all these subgroups N this means that $G_{t-1} \subseteq N$ for all $N \in \mathcal{N}$, i.e. $G_{t-1} \subseteq \bigcap_{N\in\mathcal{N}} N$ and, therefore, the intersection $\bigcap_{N\in\mathcal{N}} N$ has finite index in G. The finiteness of the index implies the finiteness of the set $\mathcal{N}$, as required.

To obtain the bound it is sufficient to note that if all subgroups in $\mathcal{N}$ have index not larger than n, then

$$|G:G_k| \leqslant |G:G_{k-1}|n^{|\mathcal{N}_k|} \quad \text{and} \quad |\mathcal{N}_k| \leqslant 2^{|G:G_{k-1}|} \quad \text{(this is a very rough estimate)}.$$

Therefore,

$$|G:G_k| \leqslant |G:G_{k-1}| \cdot n^{2^{|G:G_{k-1}|}}, \quad \text{i.e.} \quad |G:G_{t-1}| \leqslant F^{t-1}(n) \quad \text{and} \quad |\mathcal{N}| \leqslant 2^{F^{t-1}(n)},$$

where $F^k(x)$ is the k-th iteration of the function $F(x) = xn^{2^x}$. $\square$

Proof of Theorem 1.1. In view of Lemma 2.2, Theorem 1.1 is a straightforward consequence of Theorem 3.3. Indeed, let the subgroup N be maximal by inclusion among all normal subgroups satisfying the hypothesis of Theorem 1.1. We set $W = W_{k-1} \circ \cdots \circ W_1 \circ W_0$, where $W_i = w_i$ if the quotient

N_i/N_{i+1} satisfies the outer commutator law $w_i \equiv 1$ and $W_i = [x, y]_*$ if the quotient N_i/N_{i+1} is locally nilpotent. By Lemma 2.2 the subgroup $W(N)$ is trivial and consequently $W(N^\alpha) = 1$ for any automorphism α of G. The subgroup

$$H = \bigcap_{\alpha \in \operatorname{Aut} G} N^\alpha$$

is characteristic. By Theorem 3.3 the index $[G : H]$ is bounded by a function depending only on $[G : N]$ and the weight of the star word W. The required series is

$$H = H_0 \geqslant H_1 \geqslant \cdots \geqslant H_k = 1,$$

where $H_i = W_{i-1}(W_{i-2}(H)) = (W_{i-1} \circ \cdots \circ W_1 \circ W_0)(H)$. In fact, all the H_i are characteristic as subgroups corresponding to some star words; H_i/H_{i+1} is locally nilpotent by Lemma 2.1 if N_i/N_{i+1} is locally nilpotent, and H_i/H_{i+1} satisfies the outer commutator law $w_i \equiv 1$ if N_i/N_{i+1} satisfies the same commutator law $w_i \equiv 1$. $\square$

We can also adapt the proof of Klyachko and Melnikova [5] to obtain a better bound for the index.

Theorem 3.4. *Let $W(x_1, \ldots, x_t)$ be a star word. If a group G contains a normal finite-index subgroup N such that $W(N) = 1$, then G contains a characteristic and even invariant with respect to all surjective endomorphisms subgroup H such that $W(H) = 1$ and $\log_2|G : H| \leqslant f^{t-1}(\log_2|G : N|)$, where $f^k(x)$ is the k-th iteration of the function $f(x) = x(x+1)$.*

Proof. The proof is virtually the same as the proof in [5]: we should only replace the outer commutator $\omega(x_1, \ldots, x_n)$, which defines the outer commutator law $w \equiv 1$ in the hypothesis of theorem in [5], by the star word $W(x_1, \ldots, x_t)$ and use Corollary 3.2 instead of the similar property for the ordinary commutators subgroups and outer commutator words. $\square$

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